Integrability of V. Adler's discretization of the Neumann system

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Abstract

We prove the integrability of the discretization of the Neumann system recently proposed by V. Adler.

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1 Introduction

The famous Neumann system belongs to the area of constrained mechanics. It describes the motion of a particle on the surface of the sphere $S = \{x \in \mathbb{R}^N : \langle x, x\rangle = 1\}$ under the influence of the harmonic potential $\frac{1}{2}\langle \Omega x, x\rangle$, where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$. (Here and below $\langle \cdot, \cdot\rangle$ stands for the standard Euclidean scalar product in $\mathbb{R}^N$). The equations of motion read:

$$\ddot{x}_k = -\omega_k x_k - \alpha x_k\ , \qquad 1 \le k \le N\ , \tag{1}$$

where $\alpha = \alpha(x, \dot{x})$ is the Lagrange multiplier assuring the validity of the relations $\langle x, x\rangle = 1$, $\langle \dot{x}, x\rangle = 0$ during the evolution. It is easy to see that

$$\alpha = \langle \dot{x}, \dot{x}\rangle - \langle \Omega x, x\rangle\ . \tag{2}$$

This system can be given a Hamiltonian interpretation in terms of the Dirac Poisson bracket, and it turns out to be completely integrable in the Liouville–Arnold sense, all integrals being quadratic in momenta (which also implies that it is solvable via the separation of variables method). The literature on the Neumann system includes the original [1], as well as modern presentations, e.g. [2, 3, 4, 5].

The present paper is devoted to a discretization of (1), introduced recently by V. Adler in a beautiful paper [6], along with a novel discretization of the Landau–Lifschitz system. In difference equations for $x : h\mathbb{Z} \mapsto S$, we write x for $x(nh)$, $\widetilde{x}$ for $x((n+1)h)$, and $\underset{\sim}{x}$ for $x((n-1)h)$. The equations of motion of the V.Adler's discretization read:

$$\frac{\widetilde{x}_k + x_k}{1 + \langle \widetilde{x}, x\rangle} + \frac{x_k + \underset{\sim}{x}_k}{1 + \langle x, \underset{\sim}{x}\rangle} = \frac{(2 - \frac{h^2}{2}\omega_k)x_k}{1 - \frac{h^2}{4}\langle \Omega x, x\rangle}\ , \quad 1 \le k \le N\ . \tag{3}$$

While the integrability of this system for $N = 3$ follows from Adler's results, the general case was left open by him. We construct explicitly the full set of integrals of motion for (3).

It has to be said that several different integrable discretization of the Neumann system are currently known. The first one of them, due to A. Veselov [7, 8] is governed by the following difference equations:

$$\widetilde{x}_k + \underset{\sim}{x}_k = (1 + h^2\omega_k)^{-1/2}\beta x_k\ , \quad 1 \le k \le N\ , \tag{4}$$

where β is the Lagrange multiplier, assuring that x remains on the sphere S during the discrete time evolution. It is easy to see that

$$\beta = \frac{2\Big\langle (I + h^2\Omega)^{-1/2}x, \underset{\sim}{x}\Big\rangle}{\Big\langle (I + h^2\Omega)^{-1}x, x\Big\rangle} = \frac{2\Big\langle (I + h^2\Omega)^{-1/2}x, \widetilde{x}\Big\rangle}{\Big\langle (I + h^2\Omega)^{-1}x, x\Big\rangle} = \frac{\Big\langle (I + h^2\Omega)^{-1/2}x, \widetilde{x} + \underset{\sim}{x}\Big\rangle}{\Big\langle (I + h^2\Omega)^{-1}x, x\Big\rangle}\ . \tag{5}$$

The first two expressions for β make this difference equation *explicit* for the evolution in both the positive and the negative directions of (discrete) time. The third expression makes it obvious that $\beta = 2 - h^2\alpha + o(h^2)$ with α from (2), so that we indeed have a discretization of (1). This equation (4) is a discrete time Lagrangian system, which allows one to introduce the canonically conjugate momenta in a systematic

way. Performing this, the main feature of (4) comes into the light: namely, it shares the integrals of motion with its continuous time counterpart. In other words, it is a Bäcklund transformation for the continuous time Neumann system (cf. [9]).

The second integrable discretization of the Neumann system, discovered by O. Ragnisco [10], is governed by the following difference equations:

$$\frac{\widetilde{x}_k}{\langle\widetilde{x},x\rangle}-2x_k+\frac{\underset{\sim}{x}_k}{\langle x,\underset{\sim}{x}\rangle}=-h^2\omega_k x_k+h^2\langle\Omega x,x\rangle x_k\ ,\quad 1\le k\le N\ . \tag{6}$$

Again, this is a discrete time Lagrangian system. Introducing the canonically conjugate momenta in a proper way, one sees that the integrals of motion for (6) are $O(h)$–perturbations of the integrals of the continuous time Neumann system.

The comparison of the previously known discretizations was performed in [11]. With the advent of the V. Adler's discretization and the proof of its integrability, the Neumann system becomes the best candidate for the championship in possessing various different integrable discretizations.

2 Neumann system

The Neumann system describes the motion of a point $x\in\mathbb{R}^N$ under the potential $\frac{1}{2}\langle\Omega x,x\rangle$, constrained to the sphere

$$S=\Big\{x\in\mathbb{R}^N:\ \langle x,x\rangle=1\Big\}. \tag{1}$$

The Lagrangian approach to this problem is as follows. The motions should deliver local extrema to the action functional

$$\mathbf{S}=\int_{t_0}^{t_1}\mathbf{L}\Big(x(t),\dot{x}(t)\Big)dt\ ,$$

where $\mathbf{L}:TS\mapsto\mathbb{R}$ is the Lagrange function, given by

$$\mathbf{L}(x,\dot{x})=\frac{1}{2}\langle\dot{x},\dot{x}\rangle-\frac{1}{2}\langle\Omega x,x\rangle-\frac{1}{2}\,\alpha(\langle x,x\rangle-1)\ . \tag{2}$$

Here the first two terms on the right–hand side represent the unconstrained Lagrange function, and the Lagrange multiplier α has to be chosen to assure that the solution of the variational problem lies on the constrained manifold TS, which is described by the equations

$$\langle x,x\rangle=1\ ,\qquad\langle\dot{x},x\rangle=0\ . \tag{3}$$

The differential equations of the extremals of the above problem read:

$$\ddot{x}_k=-\omega_k x_k-\alpha x_k\ , \tag{4}$$

or, in the vector form,

$$\ddot{x}=-\Omega x-\alpha x\ . \tag{5}$$

The value of α is determined as follows:

$$0=\langle\dot{x},x\rangle^{\bullet}=\langle\dot{x},\dot{x}\rangle+\langle\ddot{x},x\rangle=\langle\dot{x},\dot{x}\rangle-\langle\Omega x,x\rangle-\alpha\ .$$

Therefore

$$\alpha = \langle \dot{x}, \dot{x} \rangle - \langle \Omega x, x \rangle \ . \tag{6}$$

So, the complete description of the Neumann problem is delivered by the equations of motion (4), augmented with the expression (6).

The Legendre transformation, leading to the Hamiltonian interpretation of the above system, is given by:

$$\mathrm{H}(x,p) = \langle \dot{x}, p \rangle - \mathbf{L}(x, \dot{x}) \ ,$$

where the canonically conjugate momenta p are given by

$$p = \partial \mathrm{L} / \partial \dot{x} = \dot{x} \ . \tag{7}$$

Hence we obtain

$$\mathrm{H}(x,p) = \frac{1}{2}\langle p, p \rangle + \frac{1}{2}\langle \Omega x, x \rangle \ . \tag{8}$$

The corresponding symplectic structure is the restriction of the standard symplectic structure of the space $\mathbb{R}^{2N}(x,p)$,

$$\{p_k, x_j\} = \delta_{kj} \ , \tag{9}$$

to the submanifold T^*S which is singled out by the relations

$$\phi_1 = \langle x, x \rangle - 1 = 0 \ , \qquad \phi_2 = \langle p, x \rangle = 0 \ . \tag{10}$$

The Dirac Poisson bracket for this symplectic structure on T^*S is given by the following relations:

$$\{x_k, x_j\}_{\mathrm{D}} = 0 \ , \quad \{p_k, x_j\}_{\mathrm{D}} = \delta_{kj} - \frac{x_k x_j}{\langle x, x \rangle} \ , \quad \{p_k, p_j\}_{\mathrm{D}} = \frac{x_k p_j - p_k x_j}{\langle x, x \rangle} \ . \tag{11}$$

It is easy to check that the Hamiltonian vector field generated by the Hamilton function (8) in the bracket $\{\cdot,\cdot\}_{\mathrm{D}}$ on T^*S, is given by:

$$\dot{x}_k = \{\mathrm{H}, x_k\}_{\mathrm{D}} = p_k \ , \qquad \dot{p}_k = \{\mathrm{H}, p_k\}_{\mathrm{D}} = -\omega_k x_k - \alpha x_k \ , \tag{12}$$

where

$$\alpha = \langle p, p \rangle - \langle \Omega x, x \rangle \ . \tag{13}$$

Obviously, this is nothing but the first–order form of (4) with the multiplier (6).

Supposing that all ω_k are distinct (which will be assumed from now on), one can prove that the following N functions are integrals of motion of (12):

$$\mathrm{F}_k = x_k^2 + \sum_{j \neq k} \frac{(p_k x_j - x_k p_j)^2}{\omega_k - \omega_j} \ , \qquad 1 \le k \le N \ . \tag{14}$$

Only $N-1$ of them are functionally independent on T^*S, due to the relation

$$\sum_{k=1}^{N} \mathrm{F}_k = \langle x, x \rangle \ , \tag{15}$$

and this is equal to 1 on T^*S. The Hamilton function of the Neumann system may be represented as

$$\mathrm{H} = \frac{1}{2}\sum_{k=1}^{N} \omega_k \mathrm{F}_k = \frac{1}{2}\Big(\langle p, p \rangle \langle x, x \rangle - \langle p, x \rangle^2 \Big) + \frac{1}{2}\langle \Omega x, x \rangle \ , \tag{16}$$

which coincides with (8) on T^*S. All functions F_k are in involution, assuring the complete integrability of the Neumann system with respect to the Dirac Poisson bracket $\{\cdot,\cdot\}_{\mathrm{D}}$.

3 Adler's discretization

It will be convenient to denote

$$\alpha_k = 1 - \frac{h^2}{4}\omega_k \, , \qquad \mathrm{A} = \mathrm{diag}(\alpha_1, \ldots, \alpha_N) = I - \frac{h^2}{4}\Omega \, . \tag{17}$$

In these notations, the Adler's discretization (3) is written as

$$\frac{\widetilde{x}_k + x_k}{1 + \langle \widetilde{x}, x \rangle} + \frac{x_k + \underset{\sim}{x}_k}{1 + \langle x, \underset{\sim}{x} \rangle} = \frac{2\alpha_k x_k}{\langle \mathrm{A}x, x \rangle} \, . \tag{18}$$

From the first glance, these equations look *implicit* for the evolution in both the positive and the negative directions of time. However, this is actually not the case. Consider, for definiteness, the case of the positive time direction. We can express from (18) the quantity $1 + \langle \widetilde{x}, x \rangle$ through x and $\underset{\sim}{x}$. Indeed, rewrite (18) as

$$\frac{\widetilde{x} + x}{1 + \langle \widetilde{x}, x \rangle} = \frac{2\mathrm{A}x}{\langle \mathrm{A}x, x \rangle} - \frac{x + \underset{\sim}{x}}{1 + \langle x, \underset{\sim}{x} \rangle} \, . \tag{19}$$

Taking the square of the norm of both sides, we find:

$$\frac{2}{1 + \langle \widetilde{x}, x \rangle} = \left\| \frac{2\mathrm{A}x}{\langle \mathrm{A}x, x \rangle} - \frac{x + \underset{\sim}{x}}{1 + \langle x, \underset{\sim}{x} \rangle} \right\|^2 ,$$

With this at hand, we can use (19) to express also $\widetilde{x}$ as function of x and $\underset{\sim}{x}$.

By the way, the previous argument has an important corollary. It is easy to see that the last formula may be represented as

$$\frac{1}{1 + \langle \widetilde{x}, x \rangle} = -\frac{1}{1 + \langle x, \underset{\sim}{x} \rangle} + \frac{2\langle \mathrm{A}x, \mathrm{A}x \rangle}{\langle \mathrm{A}x, x \rangle^2} - \frac{2\langle \mathrm{A}x, \underset{\sim}{x} \rangle}{\langle \mathrm{A}x, x \rangle \, (1 + \langle x, \underset{\sim}{x} \rangle)} \, .$$

However, we could as well perform a similar calculation for the evolution in the negative direction of time, which would lead to the following formula:

$$\frac{1}{1 + \langle x, \underset{\sim}{x} \rangle} = -\frac{1}{1 + \langle \widetilde{x}, x \rangle} + \frac{2\langle \mathrm{A}x, \mathrm{A}x \rangle}{\langle \mathrm{A}x, x \rangle^2} - \frac{2\langle \mathrm{A}x, \widetilde{x} \rangle}{\langle \mathrm{A}x, x \rangle \, (1 + \langle \widetilde{x}, x \rangle)} \, .$$

Comparing the last two formulas, we come to the following conclusion:

$$\frac{\langle \mathrm{A}\widetilde{x}, x \rangle}{1 + \langle \widetilde{x}, x \rangle} = \frac{\langle \mathrm{A}x, \underset{\sim}{x} \rangle}{1 + \langle x, \underset{\sim}{x} \rangle} \, . \tag{20}$$

In other words, we found an *integral of motion* of the difference equation (18):

$$I_1(\widetilde{x}, x) = \frac{2\langle \mathrm{A}\widetilde{x}, x \rangle}{1 + \langle \widetilde{x}, x \rangle} \, . \tag{21}$$

The second integral of motion can be obtained even more straightforwardly. Indeed, from (19) there follows easily:

$$\frac{\langle \mathrm{A}^{-1}(\widetilde{x} + x), \widetilde{x} + x \rangle}{(1 + \langle \widetilde{x}, x \rangle)^2} = \frac{\langle \mathrm{A}^{-1}(x + \underset{\sim}{x}), x + \underset{\sim}{x} \rangle}{(1 + \langle x, \underset{\sim}{x} \rangle)^2} \, . \tag{22}$$

In other words, the following function is also an *integral of motion* of the difference equation (18):

$$I_2(\widetilde{x}, x) = \frac{\langle \mathrm{A}^{-1}(\widetilde{x} + x), \widetilde{x} + x \rangle}{(1 + \langle \widetilde{x}, x \rangle)^2} \, . \tag{23}$$

4 Lagrangian formulation

We now demonstrate that (18) may be interpreted as the equation of extremals of a discrete time action functional

$$\mathbb{S} = \sum_{n=n_0}^{n_1} \mathbb{L}\Big(x((n+1)h), x(nh)\Big) ,$$

where $\mathbb{L} : S \times S \mapsto \mathbb{R}$ is a discrete time Lagrange function. Recall that the discrete time Lagrangian equations of motion read:

$$\nabla_x \mathbb{L}(\widetilde{x}, x) + \nabla_x \mathbb{L}(x, \underset{\sim}{x}) = 0 , \tag{24}$$

and that the momenta $p \in T_x^* S$, $\widetilde{p} \in T_{\widetilde{x}}^* S$ canonically conjugate to x, resp. to $\widetilde{x}$, are given by:

$$\begin{aligned} hp &= -\nabla_x \mathbb{L}(\widetilde{x}, x) , & (25) \\ h\widetilde{p} &= \nabla_{\widetilde{x}} \mathbb{L}(\widetilde{x}, x) . & (26) \end{aligned}$$

Consider the discrete time Lagrange function on $S \times S$:

$$h\mathbb{L}(\widetilde{x}, x) = -2 \log(1 + \langle \widetilde{x}, x \rangle) + 2 \log\langle \mathrm{A}x, x \rangle , \quad x \in S, \ \widetilde{x} \in S . \tag{27}$$

Applying (25), (26), we find:

$$\begin{aligned} hp_k &= \frac{2\widetilde{x}_k}{1 + \langle \widetilde{x}, x \rangle} - \frac{4\alpha_k x_k}{\langle \mathrm{A}x, x \rangle} - \gamma x_k , & (28) \\ h\widetilde{p}_k &= -\frac{2x_k}{1 + \langle \widetilde{x}, x \rangle} + \delta \widetilde{x}_k . & (29) \end{aligned}$$

Here the scalar multipliers γ, δ have to be chosen so as to assure that

$$p \in T_x^* S , \qquad \widetilde{p} \in T_{\widetilde{x}}^* S ,$$

or, in other words, to assure that the following relations hold:

$$\langle p, x \rangle = 0 , \qquad \langle \widetilde{p}, \widetilde{x} \rangle = 0 . \tag{30}$$

It is easy to see that this is achieved if

$$\gamma = \frac{2\langle \widetilde{x}, x \rangle}{1 + \langle \widetilde{x}, x \rangle} - 4 = -\frac{2}{1 + \langle \widetilde{x}, x \rangle} - 2 , \qquad \delta = \frac{2\langle \widetilde{x}, x \rangle}{1 + \langle \widetilde{x}, x \rangle} = -\frac{2}{1 + \langle \widetilde{x}, x \rangle} + 2 . \tag{31}$$

So, the following are the Lagrangian equations of motion of the Adler's discretization of the Neumann system:

$$\begin{aligned} hp_k &= \frac{2(\widetilde{x}_k + x_k)}{1 + \langle \widetilde{x}, x \rangle} - \frac{4\alpha_k x_k}{\langle \mathrm{A}x, x \rangle} + 2x_k , & (32) \\ h\widetilde{p}_k &= -\frac{2(\widetilde{x}_k + x_k)}{1 + \langle \widetilde{x}, x \rangle} + 2\widetilde{x}_k . & (33) \end{aligned}$$

Clearly, these two equations yield also the Newtonian form (18) of the equations of motion.

The equations (32), (33) define a symplectic map $(x,p) \in T^*S \mapsto (\widetilde{x},\widetilde{p}) \in T^*S$ *explicitly.* Indeed, the first equation (32) yields

$$\frac{2(\widetilde{x}_k + x_k)}{1+\langle \widetilde{x}, x\rangle} = \frac{4\alpha_k x_k}{\langle \mathrm{A}x, x\rangle} - 2x_k + hp_k \ ,$$

which implies

$$\frac{2}{1+\langle \widetilde{x}, x\rangle} = \left\| \frac{2\mathrm{A}x}{\langle \mathrm{A}x, x\rangle} - x + \frac{hp}{2}\right\|^2 \ . \tag{34}$$

This, substituted back into (32), allows us to determine $\widetilde{x}$, and then, finally, the second equation of motion (33) defines $\widetilde{p}$. (Clearly, this is a perifrase of the similar argument in the previous section).

5 Integrability

We can now express the integrals of motion I_1, I_2 in terms of canonically conjugate variables (x,p). Straightforward calculations verify the following formulas:

$$I_1(\widetilde{x}, x) = \frac{2\langle \mathrm{A}\widetilde{x}, x\rangle}{1+\langle \widetilde{x}, x\rangle} = \langle \mathrm{A}x, x\rangle - h\langle \mathrm{A}x, p\rangle - \frac{h^2}{4}\langle p,p\rangle\langle \mathrm{A}x, x\rangle \ , \tag{35}$$

$$I_2(\widetilde{x}, x) = \frac{\langle \mathrm{A}^{-1}(\widetilde{x}+x), \widetilde{x}+x\rangle}{(1+\langle \widetilde{x}, x\rangle)^2} = \langle \mathrm{A}^{-1}x, x\rangle - h\langle \mathrm{A}^{-1}x, p\rangle + \frac{h^2}{4}\langle \mathrm{A}^{-1}p, p\rangle \ . \tag{36}$$

These two integrals are enough to assure the Liouville–Arnold integrability of the map (32), (33) for $N=3$. The full set of integrals in the general case is given in the following statement, which constitutes the main result of this Letter.

Theorem 1 *If all α_k's are distinct, then the following functions are integrals of motion of the Adler's discrete time Neumann system:*

$$\mathcal{F}_k = x_k^2 - hx_kp_k + \frac{h^2}{4}\sum_{j\neq k}\frac{(x_kp_j - x_jp_k)(\alpha_kx_kp_j - \alpha_jx_jp_k)}{\alpha_j - \alpha_k} \ . \tag{37}$$

Proof. Denote

$$\mathrm{X}_{kj} = x_kp_j - x_jp_k \ , \qquad \mathrm{Y}_{kj} = \alpha_kx_kp_j - \alpha_jx_jp_k \ . \tag{38}$$

We have:

$$\sum_{j\neq k}\frac{\widetilde{\mathrm{X}}_{kj}\widetilde{\mathrm{Y}}_{kj}}{\alpha_j-\alpha_k} - \sum_{j\neq k}\frac{\mathrm{X}_{kj}\mathrm{Y}_{kj}}{\alpha_j-\alpha_k} = \sum_{j\neq k}\frac{\widetilde{\mathrm{X}}_{kj}(\widetilde{\mathrm{Y}}_{kj}-\mathrm{Y}_{kj}) + \mathrm{Y}_{kj}(\widetilde{\mathrm{X}}_{kj}-\mathrm{X}_{kj})}{\alpha_j-\alpha_k} \ . \tag{39}$$

Using the equations of motion (32), (33), we find:

$$\begin{aligned}
\frac{h}{2}\widetilde{\mathrm{X}}_{ij} &= \frac{x_k\widetilde{x}_j - \widetilde{x}_kx_j}{1+\langle \widetilde{x}, x\rangle} \ , \\
\frac{h}{2}\mathrm{Y}_{kj} &= \frac{\alpha_kx_k\widetilde{x}_j - \alpha_j\widetilde{x}_kx_j + (\alpha_k-\alpha_j)x_kx_j}{1+\langle \widetilde{x}, x\rangle} + (\alpha_k-\alpha_j)x_kx_j \ ,
\end{aligned}$$

and

$$
\begin{aligned}
\frac{h}{2}\,\frac{\widetilde{\mathrm{X}}_{kj}-\mathrm{X}_{kj}}{\alpha_j-\alpha_k} &= \frac{2x_kx_j}{\langle \mathrm{A}x,x\rangle}\,,\\
\frac{h}{2}\,\frac{\widetilde{\mathrm{Y}}_{kj}-\mathrm{Y}_{kj}}{\alpha_j-\alpha_k} &= \frac{x_kx_j+\widetilde{x}_kx_j+x_k\widetilde{x}_j+\widetilde{x}_k\widetilde{x}_j}{1+\langle\widetilde{x},x\rangle}+x_kx_j-\widetilde{x}_k\widetilde{x}_j\ .
\end{aligned}
$$

Calculating further, we find:

$$
\begin{aligned}
&\frac{h^2}{4}\sum_{j\neq k}\frac{\widetilde{\mathrm{X}}_{kj}(\widetilde{\mathrm{Y}}_{kj}-\mathrm{Y}_{kj})}{\alpha_j-\alpha_k} =\\
&= \frac{1}{1+\langle\widetilde{x},x\rangle}\sum_{j=1}^{N}(x_k\widetilde{x}_j-\widetilde{x}_kx_j)\Big(\frac{x_kx_j+\widetilde{x}_kx_j+x_k\widetilde{x}_j+\widetilde{x}_k\widetilde{x}_j}{1+\langle\widetilde{x},x\rangle}+x_kx_j-\widetilde{x}_k\widetilde{x}_j\Big)\\
&= \frac{1}{1+\langle\widetilde{x},x\rangle}\Big(x_k^2(1+\langle\widetilde{x},x\rangle)-2\widetilde{x}_kx_k-\widetilde{x}_k^2(1-\langle\widetilde{x},x\rangle)\Big)\\
&= x_k^2-\widetilde{x}_k^2+h\widetilde{x}_k\widetilde{p}_k\ ;
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h^2}{4}\sum_{j\neq k}\frac{\mathrm{Y}_{kj}(\widetilde{\mathrm{X}}_{kj}-\mathrm{X}_{kj})}{\alpha_j-\alpha_k} =\\
&= \frac{2}{\langle \mathrm{A}x,x\rangle}\sum_{j=1}^{N}x_kx_j\Big(\frac{\alpha_kx_k\widetilde{x}_j-\alpha_j\widetilde{x}_kx_j+(\alpha_k-\alpha_j)x_kx_j}{1+\langle\widetilde{x},x\rangle}+(\alpha_k-\alpha_j)x_kx_j\Big)\\
&= \frac{2x_k}{\langle \mathrm{A}x,x\rangle}\Big(2\alpha_kx_k-\frac{(\widetilde{x}_k+x_k)}{1+\langle\widetilde{x},x\rangle}\langle \mathrm{A}x,x\rangle-x_k\langle \mathrm{A}x,x\rangle\Big)\\
&= -hx_kp_k\ .
\end{aligned}
$$

Collecting all results, we find:

$$
\frac{h^2}{4}\sum_{j\neq k}\frac{\widetilde{\mathrm{X}}_{kj}\widetilde{\mathrm{Y}}_{kj}}{\alpha_j-\alpha_k}-\frac{h^2}{4}\sum_{j\neq k}\frac{\mathrm{X}_{kj}\mathrm{Y}_{kj}}{\alpha_j-\alpha_k}=x_k^2-\widetilde{x}_k^2+h\widetilde{x}_k\widetilde{p}_k-hx_kp_k\ .
$$

This proves the Theorem. ■

The involutivity of the integrals $\mathcal{F}_k$ may be established by the arguments similar to those in [5]. It is easy to see that the two previously found integrals (35), (36) admit the following expression in terms of $\mathcal{F}_k$:

$$
2I_1=\sum_{k=1}^{N}\alpha_k\mathcal{F}_k\ ,\qquad I_2=\sum_{k=1}^{N}\alpha_k^{-1}\mathcal{F}_k\ .
$$

6 Conclusions

Several things remain to be done to put the V. Adler's discrete time Neumann system into the modern framework of discrete integrable systems: to find Lax representations and their r–matrix interpretation, and to perform an integration in terms of theta–functions. I hope to have an opprtunity to present these results elsewhere. I thank V. Adler and A. Shabat for showing me their results prior to publication.

References

[1] C.Neumann. De problemate quodam mechanica, quod ad primam integralium ultraellipticorum classem revocatur. *J. Reine Angew. Math.* **56** (1859) 46–69.

[2] J.Moser. Geometry of quadrics and spectral theory. In: *Chern Symposium 1979*, Springer, 1981, p.147–188.

[3] T.Ratiu. The C.Neumann problem as a completely integrable system on an adjoint orbit. *Trans. Amer. Math. Soc.* **264** (1981) 321–329.

[4] H.Flashka. Towards an algebro–geometric interpretation of the Neumann problem. *Tohoku Math. J.* **36** (1984) 407–426.

[5] D.Mamford. Tata lectures on theta. Birkhäuser, 1984.

[6] V.Adler. Discretizations of the Landau–Lifshitz system. *Teor. Math. Phys.* (2000, submitted).

[7] A.P.Veselov. Integrable systems with discrete time and difference operators. *Funct. Anal. Appl.* **22** (1988) 1–13.

[8] J.Moser, A.P.Veselov. Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Commun. Math. Phys.* **139** (1991) 217–243.

[9] A.Hone, V.Kuznetsov, O.Ragnisco. Bäcklund transformations for many-body systems related to KdV. *J. Phys. A: Math. Gen.* **32** (1999) L299–L306.

[10] O.Ragnisco. A discrete Neumann system. *Phys. Lett. A* **167** (1992) 165–171.

[11] O.Ragnisco, Yu. Suris. On the r–matrix structure of the Neumann system and its discretizations. In: *Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman*, Birkhäuser, 1996, p.285–300.